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A Dynamical Theory of Structured Solids. II Special Constitutive Equations and Special Cases of the Theory

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Phil. Trans. R. Soc. Lond. A 1993 **345**, 459-476
doi: 10.1098/rsta.1993.0141

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This paper is a continuation of Part I under the same title and is concerned with derivation of some special cases of the general theory of Part I applicable to elastic-plastic and elastic-viscoplastic single crystals. The main object here is to identify several existing macroscopic theories of inelastic material behaviour and to shed light on the range of their validity in relation to accepted notions of various physical scales associated with the motion of crystal lattice. Included among the results obtained are: (i) the identification of the elastic part of the intrinsic lattice force with the so-called 'energy–momentum tensor' using Eshelby's terminology; (ii) the development of special elastic-viscoplastic and elastic-plastic theories of material behaviour in which the inertia effect associated with the rate of plastic deformation is neglected but other microstructural effects are retained; and (iii) the reduction, within the framework of the rate-independent theory, to Prandtl–Reuss type equations in which all microstructural effects are suppressed.

Phil. Trans. R. Soc. Lond. A (1993) **345**, 459–476
Printed in Great Britain

1. Introduction

This paper is a companion to Part I under the same title (Naghdi & Srinivasa 1993). It contains a dynamical theory of structured solids that takes into account the effect of motion of dislocations and related crystal deformations at microscopic and submicroscopic scales. Although the basic theory (in §§3 and 4 of Part I) is applicable to a wide range of material behaviour, the discussion of constitutive equations (in §§5 and 6 of Part I) is focussed on fairly general developments appropriate for elastic-plastic and elastic-viscoplastic single crystals. Given this background, the main purpose of the present Part II is to consider special constitutive equations and to derive important special cases from the general theory of Part I (hereafter frequently referred to simply as I) in a manner that (i) identifies a number of existing macroscopic theories of inelastic material behaviour and (ii) clarifies the nature of their range of validity in relation to the various physical scales of motion as described in I, §2*a*.

It may be emphasized here that the general development of the material response in I, §6, is rate-dependent (or viscoplastic) in the sense that the response function $\hat{\boldsymbol{K}}$, given by I, eqn (6.18), but not for the stress $\boldsymbol{S}$ (see I, eqn $(6.12)_1$), depends on the plastic deformation rate $\dot{\boldsymbol{G}}_{\rm p}$. Some of the derived special or restricted results (in §§3–5) are obtained by utilizing an interesting particular functional form for the stress $\boldsymbol{S}$ (or the Cauchy stress $\boldsymbol{T}$), by neglecting the inertia effects due to plastic deformation and by suppressing some parts of the expression for the response $\hat{\boldsymbol{K}}$.

(*a*) *A summary description of the results obtained and the notation used*

A good idea of the coverage in Part II can be gained from an examination of the list of contents preceding the abstract. Here we highlight the nature of some of the main results as follows.

1. A special choice for the constitutive response of the specific Helmholtz free energy ψ (see (2.3) and (2.15)) which permits the identification of the elastic part of the intrinsic lattice force $\boldsymbol{K}$ with that known as the 'energy–momentum tensor' using Eshelby's (1970) terminology; in this connection, see also (2.17) and the paragraph following it.

2. Derivation of a special (rate-dependent) theory of elastic-viscoplastic behaviour by suppressing the effect of director inertia in the director momentum equations of motion. In particular, this development results in (i) an equation for the determination of the plastic deformation rate $\dot{\boldsymbol{G}}_{\rm p}$ and (ii) the loading criterion in terms of a yield function g of the kinematical variables (see $(3.11)_{1,2}$). An illustrative example at the end of §3, demonstrating the manner of calculation of $\dot{\boldsymbol{G}}_{\rm p}$ with the use of the constitutive results of §2.

3. Derivation of a special (rate-independent) theory of elastic-plastic materials in §4 by suppressing the effect of director inertia in the director momenta equations of motion, as well as that of the viscous effect in the constitutive equations of the general theory. One consequence of results is the expression (4.1) for the determination of the direction $\boldsymbol{\rho}$ of the tensor $\dot{\boldsymbol{G}}_{\rm p}$ in terms of the kinematical variables $\mathscr{U}$ defined by (I, eqn $(5.6)_1$).

4. A demonstration in §4 that the consistency condition (4.4), which must be satisfied during plastic deformation, is both necessary and sufficient for the existence of a solution of (4.1) as an equation for the determination of $\boldsymbol{\rho}$.

5. A discussion of the loading criteria in the context of a (rate-independent) theory

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of elastic-plastic materials which shows that the necessity for having loading criteria is intimately related to the determination of the magnitude γ of $\dot{\boldsymbol{G}}_{\mathrm{p}}$ as a function of the variables $\mathscr{U}$. The expression (4.12) for $\hat{g}$, along with (4.20) and (4.9) constitute the loading criteria.

6. A procedure for the determination of γ from (4.11) which leads to (4.18) and a solution of the form (4.19). However, an explicit determination of γ from (4.19) is rather intricate due to the dependence of g on grad $\boldsymbol{G}_{\mathrm{p}}$: This is because of the presence of $\hat{g}$ (as one of the integrands) in (4.19). In this connection, it should be recalled that grad $\boldsymbol{G}_{\mathrm{p}}$ represents a measure of the lattice defects that provides a connection between changes in dislocation density to processes involving plastic deformation (I, §3*c*).

7. Reduction to the standard loading criteria of plasticity (see Naghdi 1990, eqns (4.18)) when the dependence on grad $\boldsymbol{G}_{\mathrm{p}}$ is suppressed. (Actually, even in the absence of dependence of the yield function on grad $\boldsymbol{G}_{\mathrm{p}}$, due to the nonsymmetric part of the tensor $\boldsymbol{G}_{\mathrm{p}}$ the reduced $\hat{g}$ in the form given by (4.12) is still more general than the corresponding expression in the standard theory of elastic-plastic materials.)

8. Reduction of the rate-independent theory of §4 to Prandtl–Reuss type equations (in §5*b*) with the use of the linearized version of the constitutive equations of §2 along with the suppression of all microstructural effects by invoking the conditions $\boldsymbol{\beta} = \boldsymbol{0}$ and $\kappa = \text{const.}$, where $\boldsymbol{\beta}$ and κ are defined by (2.19).

Before closing this section, it is desirable to make some comments regarding the notations used. The notation here is the same as in Part I and uses mainly a direct tensor notation (see I, §1*b*). However, in a number of places in Part II it appeared to be less cumbersome to record certain expressions and tensor identities in their component forms (rather than in direct notation), as, for example, in (2.4) and (5.14). Further, it should be emphasized that in the remainder of Part II, whenever reference is made to an equation such as (6.18) of Part I, this is indicated as (I, eqn (6.18)).

2. Special forms for constitutive equations

We specialize in this section the general constitutive developments of §6 of Part I and begin by introducing a new set of independent variables $\bar{\mathscr{U}}$ defined by

$$\bar{\mathscr{U}} = ({}_{\ell}\boldsymbol{F}, \mathscr{W}), \tag{2.1}$$

where ${}_{\ell}\boldsymbol{F}$ is the lattice deformation tensor defined by (I, eqn $(3.6)_1$) and $\mathscr{W}$ stands for the set of variables (I, eqn $(5.6)_2$). The set of kinematical variables $\mathscr{U}$ defined by (I, eqn $(5.6)_1$) can be regarded as a function of the variables (2.1). For this purpose, we observe from the relationship (I, eqn (3.8)) between $(\boldsymbol{F}, {}_{\ell}\boldsymbol{F}, \boldsymbol{G}_{\mathrm{p}})$ that $\boldsymbol{F}$ (and hence also the lagrangian strain $\boldsymbol{E}$) may be regarded as a function of $({}_{\ell}\boldsymbol{F}, \boldsymbol{G}_{\mathrm{p}})$, so that the variables $\mathscr{U}$ may be obtained from $\bar{\mathscr{U}}$ through a functional relationship of the form

$$\mathscr{U} = \{\hat{E}({}_{\ell}\boldsymbol{F}, \boldsymbol{G}_{\mathrm{p}}), \mathscr{W}\} = \hat{\mathscr{U}}({}_{\ell}\boldsymbol{F}, \mathscr{W}) = \hat{\mathscr{U}}(\bar{\mathscr{U}}), \tag{2.2}$$

where in recording $(2.2)_1$ we have recalled the definition of $\boldsymbol{E}$ (I, eqns $(3.26)_1$ and $(3.25)_1$), substituted $\boldsymbol{F} = {}_{\ell}\boldsymbol{F}\boldsymbol{G}_{\mathrm{p}}$ (see I, eqn (3.8)) and used the temporary notation $\hat{\boldsymbol{E}}({}_{\ell}\boldsymbol{F}, \boldsymbol{G}_{\mathrm{p}}) = \frac{1}{2}[({}_{\ell}\boldsymbol{F}\boldsymbol{G}_{\mathrm{p}})^{\mathrm{T}}\,{}_{\ell}\boldsymbol{F}\boldsymbol{G}_{\mathrm{p}} - \boldsymbol{I}]$.

In view of the relationship (2.2), any function of $\mathscr{U}$ may be expressed as a different function of $\bar{\mathscr{U}}$. In particular, the constitutive assumption (I, eqn (6.6)) can also be rewritten as

$$\psi = \hat{\psi}(\mathscr{U}) = \hat{\psi}[\hat{\mathscr{U}}(\bar{\mathscr{U}})] = \bar{\psi}(\bar{\mathscr{U}}). \tag{2.3}$$

Now, with the help of identities (stated here in component forms referred to the orthonormal basis $\boldsymbol{E}_A$)

$$\partial\, {}_{\ell}F_{iA}/\partial F_{jB} = \delta_{ij}(G^{\mathrm{p}})^{-1}_{BA}, \quad \partial\, {}_{\ell}F_{iA}/\partial G^{\mathrm{p}}_{CD} = -\,{}_{\ell}F_{iC}(G^{\mathrm{p}})^{-1}_{DA} \tag{2.4}$$

and

$$\partial \hat{E}_{AB}/\partial({}_{\ell}F_{jC}) = \tfrac{1}{2}[G^{\mathrm{p}}_{CA}F_{jB} + G^{\mathrm{p}}_{CB}F_{jA}] \tag{2.5}$$

and using the chain rule, the constitutive results (I, eqn $(6.12)_{1,2}$) and the rate-independent response $\hat{\boldsymbol{K}}_1$ defined by (I, eqn (6.15)) can be rewritten in terms of $\bar{\psi}$ in (2.3) as follows:

$$\left.\begin{aligned} \boldsymbol{S} &= \rho_0 \boldsymbol{F}^{-1}\frac{\partial\bar{\psi}}{\partial\,{}_{\ell}\boldsymbol{F}}\boldsymbol{G}_{\mathrm{p}}^{-\mathrm{T}} = \bar{S}(\bar{\mathscr{U}}), \quad {}_{\mathrm{R}}\mathscr{M} = \rho_0\frac{\partial\bar{\psi}}{\partial\,\mathrm{grad}\,\boldsymbol{G}_{\mathrm{p}}}, \\ \hat{\boldsymbol{K}}_1 &= \rho_0\left[\frac{\partial\bar{\psi}}{\partial \boldsymbol{G}_{\mathrm{p}}} - {}_{\ell}\boldsymbol{F}^{\mathrm{T}}\frac{\partial\bar{\psi}}{\partial\,{}_{\ell}\boldsymbol{F}}\boldsymbol{G}_{\mathrm{p}}^{-\mathrm{T}}\right]. \end{aligned}\right\} \tag{2.6}$$

Now the Piola–Kirchhoff stress tensor $\boldsymbol{P}$ which occurs in the equations of motion (I, eqn $(4.9)_2$), as well as the Cauchy stress tensor $\boldsymbol{T}$ calculated from the well-known formula,

$$\boldsymbol{T} = J^{-1}\boldsymbol{P}\boldsymbol{F}^{\mathrm{T}}, \tag{2.7}$$

may also be expressed in terms of the variable $\bar{\mathscr{U}}$. Carrying out the necessary calculations and making use of the results $(2.6)_{1,3}$ and (I, eqn $(4.16)_1$), we arrive at the following results:

$$P = \rho_0\frac{\partial\bar{\psi}}{\partial\,{}_{\ell}\boldsymbol{F}}\boldsymbol{G}_{\mathrm{p}}^{-\mathrm{T}}, \quad \boldsymbol{T} = \frac{\rho_0}{J}\frac{\partial\bar{\psi}}{\partial\,{}_{\ell}\boldsymbol{F}}\,{}_{\ell}\boldsymbol{F}^{\mathrm{T}} = \frac{\rho_0}{J_{\ell}J_{\mathrm{p}}}\frac{\partial\bar{\psi}}{\partial\,{}_{\ell}\boldsymbol{F}}\,{}_{\ell}\boldsymbol{F}^{\mathrm{T}}, \tag{2.8}$$

where we have also introduced the notations

$$J_{\ell} = \det({}_{\ell}\boldsymbol{F}) \neq 0, \quad J_{\mathrm{p}} = \det \boldsymbol{G}_{\mathrm{p}} \neq 0. \tag{2.9}$$

(a) *Special constitutive equations for the rate-independent terms in the overall response of material*

With the help of the various results between (2.1)–(2.9), we consider in this subsection special constitutive equations which lead to interesting forms for the *rate-independent* terms $\boldsymbol{S}$, ${}_{\mathrm{R}}\mathscr{M}$ and $\hat{\boldsymbol{K}}_1$ in (I, eqn (6.18)). Motivated by the idealized notion that the contact forces are solely due to deformations of the lattice structure, we assume that the Cauchy stress tensor $\boldsymbol{T}$ depends only on the lattice deformation and write

$$\boldsymbol{T} = \hat{\boldsymbol{T}}({}_{\ell}\boldsymbol{F}). \tag{2.10}$$

In view of the relations $(2.8)_2$ between ψ and $\boldsymbol{T}$, the constitutive assumption (2.10) implies a restriction on the form of the function $\bar{\psi}$. For the purpose of identifying this restriction, we define a scalar function $\tilde{\psi}(\bar{\mathscr{U}})$ by

$$\tilde{\psi}(\bar{\mathscr{U}}) = J_{\mathrm{p}}^{-1}\bar{\psi}(\bar{\mathscr{U}}) \tag{2.11}$$

and then use (2.10) and the second of $(2.8)_2$ to obtain

$$\hat{\boldsymbol{T}}({}_{\ell}\boldsymbol{F}) = \rho_0 J_{\ell}^{-1}\frac{\partial\tilde{\psi}}{\partial\,{}_{\ell}\boldsymbol{F}}\,{}_{\ell}\boldsymbol{F}^{\mathrm{T}}. \tag{2.12}$$

The last result, after premultiplying by J_ℓ and then postmultiplying by ${}_\ell\boldsymbol{F}^{-\mathrm{T}}$, becomes

$$\rho_0\, \partial\tilde{\psi}/\partial\, {}_\ell\boldsymbol{F} = J_\ell\, \hat{\boldsymbol{T}}({}_\ell\boldsymbol{F})\, {}_\ell\boldsymbol{F}^{-\mathrm{T}}. \tag{2.13}$$

Observing that the right-hand side of (2.13) depends only on ${}_\ell\boldsymbol{F}$ and is independent of the set of variables $\mathscr{W}$ defined by (I, eqn $(5.6)_2$), integration of (2.13) with respect to ${}_\ell\boldsymbol{F}$ at once yields

$$\tilde{\psi}(\bar{\mathscr{U}}) = \tilde{\psi}_1({}_\ell\boldsymbol{F}) + \tilde{\psi}_2(\mathscr{W}) \tag{2.14}$$

and hence with the use of (2.11) we also have

$$\psi = \bar{\psi}(\bar{\mathscr{U}}) = J_{\mathrm{p}}[\tilde{\psi}_1({}_\ell\boldsymbol{F}) + \tilde{\psi}_2(\mathscr{W})]. \tag{2.15}$$

An examination of $(2.6)_1$ and $(2.8)_{1,2}$ easily reveals that only $\tilde{\psi}_1({}_\ell\boldsymbol{F})$ in (2.15) contributes to the three stress tensors $(\boldsymbol{S}, \boldsymbol{P}, \boldsymbol{T})$, while both $\tilde{\psi}_1({}_\ell\boldsymbol{F})$ and $\tilde{\psi}_2(\mathscr{W})$ contribute to the response $\hat{\boldsymbol{K}}_1$. However, if we make the further assumption that $\tilde{\psi}_2 = 0$, then the expressions $(2.6)_{3,2}$ reduce to

$$\hat{\boldsymbol{K}}_1 = \rho_0\, \tilde{\psi}_1\, \partial J_{\mathrm{p}}/\partial \boldsymbol{G}_{\mathrm{p}} - {}_\ell\boldsymbol{F}^{\mathrm{T}}\boldsymbol{P}, \quad {}_{\mathrm{R}}\mathscr{M} = \boldsymbol{0}, \tag{2.16}$$

where in obtaining $(2.16)_1$ use has also been made of $(2.8)_1$ and (2.11). With the help of the identity $(\partial J_{\mathrm{p}}/\partial \boldsymbol{G}_{\mathrm{p}}) = J_{\mathrm{p}}\, \boldsymbol{G}_{\mathrm{p}}^{-\mathrm{T}}$ and use of (I, eqn (3.8)) between $(\boldsymbol{G}_{\mathrm{p}}, {}_\ell\boldsymbol{F}, \boldsymbol{F})$, the expression $(2.16)_1$ can be rewritten as

$$\left.\begin{aligned} \hat{\boldsymbol{K}}_1 &= {}_\ell\boldsymbol{F}^{\mathrm{T}}(J_{\mathrm{p}}\,\rho_0\,\tilde{\psi}_1\,\boldsymbol{I} - \boldsymbol{P}\boldsymbol{F}^{\mathrm{T}})\,\boldsymbol{F}^{-\mathrm{T}}, \\ \text{or} \qquad \hat{\boldsymbol{K}}_1 &= {}_\ell\boldsymbol{F}^{\mathrm{T}}(\rho_0\,\bar{\psi}\boldsymbol{I} - \boldsymbol{P}\boldsymbol{F}^{\mathrm{T}})\,\boldsymbol{F}^{-\mathrm{T}} = \bar{\psi}\boldsymbol{G}_{\mathrm{p}}^{-\mathrm{T}} - {}_\ell\boldsymbol{C}\boldsymbol{G}_{\mathrm{p}}\,\boldsymbol{S}, \end{aligned}\right\} \tag{2.17}$$

where we have also made use of (2.3).

Remembering from (I, eqn (6.18)) that $\hat{\boldsymbol{K}}_1$ is only a part of the response function $\hat{\boldsymbol{K}}$, it is perhaps interesting to indicate here that the quantity in the parentheses on the right-hand side of (2.17) corresponds to Eshelby's (1970, pp. 83–92) 'energy–momentum tensor,' the integral of which over a surface enclosing a defect is the elastic force acting on a defect (Eshelby 1970, p. 85, eqn (18)). Our development leading to the special form (2.15) and hence also $(2.16)_1$ is influenced by a paper of Epstein & Maugin (1990), who (from a different starting point) have derived an expression analogous to (2.17) in the context of purely elastic deformation of a material with inhomogeneities in its reference state.

(*b*) *Special forms for the rate-dependent terms in the overall response of material*

We now focus attention on the *rate-dependent* terms $\hat{\boldsymbol{K}}_2$ and $\hat{\boldsymbol{K}}_3$ in the response function $\hat{\boldsymbol{K}}$ in (I, eqn (6.18)) and stipulate the special forms

$$\left.\begin{aligned} \hat{K}_2 &= \bar{\boldsymbol{K}}_2(\bar{\mathscr{U}}, \boldsymbol{\rho}) = {}_\ell\boldsymbol{C}\boldsymbol{G}_{\mathrm{p}}(\kappa\boldsymbol{\rho} + \boldsymbol{\beta}) - \psi\boldsymbol{G}_{\mathrm{p}}^{-\mathrm{T}}, \\ \hat{K}_3 &= \bar{K}_3(\bar{\mathscr{U}}, \gamma, \boldsymbol{\rho}) = \mu\, {}_\ell\boldsymbol{C}\boldsymbol{G}_{\mathrm{p}}\,\dot{\boldsymbol{G}}_{\mathrm{p}}. \end{aligned}\right\} \tag{2.18}$$

In (2.18), the second-order tensor $\boldsymbol{\beta}$ and the scalar κ depend on $\mathscr{W}$, i.e.

$$\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}(\mathscr{W}), \quad \kappa = \hat{\kappa}(\mathscr{W}), \tag{2.19}$$

μ is a scalar rate-dependent coefficient analogous to a coefficient of viscosity in viscous fluid flow, the variables $\bar{\mathscr{U}}$ are defined by (2.1) and $\mathscr{W}$ stands for the set of variables (I, eqn $(5.6)_2$). In the terminology of classical plasticity, κ in $(2.19)_2$ may be identified as the 'strain-hardening parameter' and $\boldsymbol{\beta}$ in $(2.19)_1$ may be referred to the 'back-stress' tensor.

It can be easily demonstrated that in the absence of any jump in $\ddot{\boldsymbol{G}}_\mathrm{p}$ the response $\bar{K}_2$ in $(2.18)_1$ satisfies the relation (see I, eqn (6.32))

$$\Phi(\boldsymbol{K}_2, \mathscr{U}) = [({}_{\ell}\boldsymbol{C}\boldsymbol{G}_\mathrm{p})^{-1}(\mathscr{K}_2+\psi\boldsymbol{G}_\mathrm{p}^{-\mathrm{T}})-\boldsymbol{\beta}]\cdot[({}_{\ell}\boldsymbol{C}\boldsymbol{G}_\mathrm{p})^{-1}(\bar{\boldsymbol{K}}_2+\psi\boldsymbol{G}_\mathrm{p}^{-\mathrm{T}})-\boldsymbol{\beta}] = \kappa^2,$$

where in obtaining the above result we have solved $(2.18)_2$ for $\kappa\boldsymbol{\rho}$ and have then taken the inner product of the resulting expression with itself. Further, the left-hand side of the above relation can be expressed in a quadratic form of the type

$$\Phi(\boldsymbol{K}_2, \mathscr{U}) = \boldsymbol{A}[\bar{\boldsymbol{K}}_2-\boldsymbol{B}]\cdot\boldsymbol{A}[\bar{K}_2-B] = [\bar{K}_2-\boldsymbol{B}]\,\boldsymbol{A}^\mathrm{T}\boldsymbol{A}[\bar{K}_2-B],$$

or equivalently

$$\Phi(\bar{K}_2, \mathscr{U}) = ({}_{\ell}\boldsymbol{C}\boldsymbol{G}_\mathrm{p})^{-1}[\bar{\boldsymbol{K}}_2-({}_{\ell}\boldsymbol{C}\boldsymbol{G}_\mathrm{p}\boldsymbol{\beta}-\psi\boldsymbol{G}_\mathrm{p}^{-\mathrm{T}}]\cdot({}_{\ell}\boldsymbol{C}\boldsymbol{G}_\mathrm{p})^{-1}[\bar{\boldsymbol{K}}_2-({}_{\ell}\boldsymbol{C}\boldsymbol{G}_\mathrm{p}\boldsymbol{\beta}-\psi\boldsymbol{G}_\mathrm{p}^{-\mathrm{T}}] = \kappa^2. \tag{2.20}$$

For fixed values of $\bar{\mathscr{U}}$, the above equation represents an ellipsoidal surface in K-space centred at $({}_{\ell}\boldsymbol{C}\boldsymbol{G}_\mathrm{p}\boldsymbol{\beta}-\psi\boldsymbol{G}_\mathrm{p}^{-\mathrm{T}})$. Also, in view of (I, eqn (6.32)), it is clear that the yield function in $\boldsymbol{K}$-space, i.e. the boundary of the elastic range, has the same functional form as Φ in (2.20) with $\bar{K}_2$ replaced by $({}_{\mathrm{R}}\boldsymbol{K})_{\mathrm{ind}}$. Using the result $(2.16)_1$ and the constitutive equation (2.17) for $\hat{\boldsymbol{K}}_1$, the yield function on the left-hand side of (I, eqn (5.11)) can be expressed as

$$\Phi((\boldsymbol{K}_\mathrm{R})_{\mathrm{ind}}, \mathscr{U}) = \Phi(-\hat{\boldsymbol{K}}_1(\mathscr{U}), \mathscr{U}) = (\boldsymbol{S}-\boldsymbol{\beta})\cdot(\boldsymbol{S}-\boldsymbol{\beta})-\kappa^2 = f(\boldsymbol{S}, \mathscr{W}), \tag{2.21}$$

where in obtaining $(2.21)_3$ in terms of $\boldsymbol{S}$ we also used the relationship (I, eqn (3.8)) between $(\boldsymbol{F}, {}_{\ell}\boldsymbol{F}, \boldsymbol{G}_\mathrm{p})$, as well as (I, eqn $(4.14)_1$) between the Piola–Kirchhoff stresses $(\boldsymbol{P}, \boldsymbol{S})$.

In view of the symmetry of $\boldsymbol{S}$, the yield function (2.21) can be rewritten as

$$f(\boldsymbol{S}, \mathscr{W}) = (\boldsymbol{S}-\boldsymbol{\beta}_{\mathrm{sym}})\cdot(\boldsymbol{S}-\boldsymbol{\beta}_{\mathrm{sym}})-[\kappa^2-\boldsymbol{\beta}_{\mathrm{skew}}\cdot\boldsymbol{\beta}_{\mathrm{skew}}] = 0. \tag{2.22}$$

The above equation represents a hypersurface in S-space with centre at $\boldsymbol{\beta}_{\mathrm{sym}}$ and radius $[\kappa^2-\boldsymbol{\beta}_{\mathrm{skew}}\cdot\boldsymbol{\beta}_{\mathrm{skew}}]^{\frac{1}{2}}$.

3. A special rate-dependent theory

Before embarking on the main objective of this section, it is desirable to provide some background information pertinent to the evolution in time of plastic deformation $\boldsymbol{G}_\mathrm{p}(t)$ in line with the procedure described in §6*c* of Part I. (Although $\boldsymbol{G}_\mathrm{p}$, as well as other basic variables, depend on both position and time, temporarily we display only their dependence on t.) To this end, we first observe that the balance law (I, eqn (4.12)) – even after substitution of the relevant constitutive equations from §6 of Part I – may be regarded as an equation for determination of $\boldsymbol{G}_\mathrm{p}$ which involves second partial derivative of $\boldsymbol{G}_\mathrm{p}$ with respect to t. We suppose that the initial values of $\boldsymbol{G}_\mathrm{p}$ are specified at $t = t_0$. Then, once the criterion for loading at the onset of plastic deformation (I, eqn (6.37)) is satisfied, the values of $\boldsymbol{G}_\mathrm{p}(t_0)$ and $\dot{\boldsymbol{G}}_\mathrm{p}(t_0)$ can be used in the constitutive equations (I, eqns $(6.12)_2$, (6.15) and (6.18)) to determine the initial values of ${}_{\mathrm{R}}\boldsymbol{K}$ and ${}_{\mathrm{R}}\mathscr{M}$ at $t = t_0$. These values, namely ${}_{\mathrm{R}}\boldsymbol{K}(t_0)$ and ${}_{\mathrm{R}}\mathscr{M}(t_0)$ can then be used in (I, eqn (4.12)) to determine $\ddot{\boldsymbol{G}}_\mathrm{p}(t_0)$ which subsequently determines $\dot{\boldsymbol{G}}_\mathrm{p}$ at the next instant of time.

What has been described in the preceding paragraph is a procedure for the determination of $\boldsymbol{G}_{\mathrm{p}}$ within the framework of the general rate-dependent theory of Part I. However, by introducing a certain simplifying assumption, the general theory can yield a simpler theoretical setting appropriate for an important class of application. This simplification, which will be discussed in the remainder of this section, entails suppressing the components of the inertia coefficient $\boldsymbol{Y}$ in (I, eqn (4.4)) and using the special constitutive equations developed in §2.

With the inertia tensor $\boldsymbol{Y}$ set equal to zero, the term involving $\ddot{\boldsymbol{G}}_{\mathrm{p}}$ vanishes in the director momenta equations (I, eqn (4.12)) which after substitution of (I, eqns $(6.7)_1$ and $(6.12)_2$) results in a first-order partial differential equation in time and the reference position in the form

$$\hat{\boldsymbol{K}}(\mathscr{U},\gamma,\boldsymbol{\rho}) = \bar{\boldsymbol{L}}. \tag{3.1}$$

Since this equation involves only $\dot{\boldsymbol{G}}_{\mathrm{p}}(=\gamma\boldsymbol{\rho})$, the specification of $\dot{\boldsymbol{G}}_{\mathrm{p}}(t_0)$ as an initial condition is not needed. To elaborate, we first observe that (3.1) furnishes nine scalar equations for the scalar variable γ and the eight independent component of the unit tensor $\boldsymbol{\rho}$. Now recalling (I, eqn (6.18)), (3.1) can be rewritten as

$$\hat{\boldsymbol{K}}_2(\mathscr{U},\boldsymbol{\rho})+\hat{\boldsymbol{K}}_3(\mathscr{U},\gamma,\boldsymbol{\rho}) = -\hat{\boldsymbol{K}}_1(\mathscr{U})+\bar{\boldsymbol{L}}. \tag{3.2}$$

Focussing attention on the left-hand side of (3.2), which temporarily will be designated by $\hat{\boldsymbol{K}}^*$, we may assign a geometrical interpretation to

$$\hat{\boldsymbol{K}}^*(\mathscr{U},\gamma,\boldsymbol{\rho}) = \hat{\boldsymbol{K}}_2(\mathscr{U},\boldsymbol{\rho})+\hat{\boldsymbol{K}}_3(\mathscr{U},\gamma,\boldsymbol{\rho}), \tag{3.3}$$

by a procedure similar to that adopted in §6*b* of Part I. Thus for fixed values of $(\mathscr{U}, \gamma)$, the function $\hat{\boldsymbol{K}}^*$ represents a mapping from a unit sphere in the nine-dimensional euclidean $\dot{\boldsymbol{G}}_{\mathrm{p}}$-space into the nine-dimensional $\boldsymbol{K}^*$-space. The range $\hat{\boldsymbol{K}}^*$ may then be regarded as an eight-dimensional hypersurface in $\boldsymbol{K}^*$-space. (Actually this geometrial interpretation is also valid for the general dynamical theory in Part I.) Hence, we may admit the existence of a function $\boldsymbol{\Phi}^*(\hat{\boldsymbol{K}}^*;\mathscr{U},\gamma)$ such that the equation

$$\boldsymbol{\Phi}^*(\boldsymbol{K}^*;\mathscr{U},\gamma) = 0 \tag{3.4}$$

for fixed values of $(\mathscr{U},\gamma)$ represents a hypersurface $\partial\mathscr{K}^*$ of dimension eight in the nine-dimensional euclidean $\boldsymbol{K}^*$-space; the values of $\boldsymbol{K}^*$ which lie on $\partial\mathscr{K}^*$ are all elements of the range of $\hat{\boldsymbol{K}}^*$. We also note that, as a result of the condition (I, eqn (6.19)), $\hat{\boldsymbol{K}}^*$ becomes identical to $\hat{\boldsymbol{K}}_2$ and hence the function $\boldsymbol{\Phi}^*$ coincides with the function $\boldsymbol{\Phi}_2$ (see I, eqn (6.23)) as $\gamma\to 0$.

In view of (3.2), we substitute from (3.3) into (3.4) to obtain

$$\boldsymbol{\Phi}^*(\bar{\boldsymbol{L}}-\hat{\boldsymbol{K}}_1(\mathscr{U});\mathscr{U},\gamma) = g^*(\mathscr{U},\gamma;\bar{\boldsymbol{L}}) = 0. \tag{3.5}$$

The condition $(3.5)_2$, which must be satisfied when plastic deformation occurs, provides a scalar equation for determination of the magnitude γ of the plastic deformation rate. Moreover, $(3.5)_2$ may be regarded as the analogue of the consistency condition in the usual formulations of plasticity or viscoplasticity.

Remembering that $\boldsymbol{\Phi}^*$ tends to $\boldsymbol{\Phi}_2$ as γ tends to zero and recalling also the identification of the functions $\boldsymbol{\Phi}_2$ and $\boldsymbol{\Phi}$ in (I, eqn (6.32)), it can be seen that as γ tends to zero the function g^* in $(3.5)_2$ reduces to the yield function g in strain space, i.e.

$$\begin{aligned} g^*(\mathscr{U},0;\bar{\boldsymbol{L}}) &= \boldsymbol{\Phi}^*(\bar{\boldsymbol{L}}-\hat{\boldsymbol{K}}_1(\mathscr{U});\mathscr{U},0) \\ &= \boldsymbol{\Phi}(\bar{\boldsymbol{L}}-\hat{\boldsymbol{K}}_1(\mathscr{U});\mathscr{U}) = g(\mathscr{U};\bar{\boldsymbol{L}}), \end{aligned} \tag{3.6}$$

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where in obtaining (3.6) use has been made of (I, eqns (5.11) and (5.12)).

The condition $(3.5)_2$ namely $g^*(\mathscr{U},\gamma;\bar{\boldsymbol{L}})=0$, represents sucessive yield surfaces in strain space for increasing values of γ. Moreover, in accordance with the accepted notion that the innermost of the yield surfaces in the rate-dependent theory is that associated with the yield surface in the rate-independent theory, we assume that $g(\mathscr{U};\bar{\boldsymbol{L}})=0$ with g being the function specified by $(3.6)_3$. It then follows that if for given values of $(\mathscr{U};\bar{\boldsymbol{L}})$ the condition $(3.5)_2$ is satisfied for some $\gamma>0$, then we have

$$g(\mathscr{U};\bar{\boldsymbol{L}})>0 \tag{3.7}$$

for the same values of $(\mathscr{U};\bar{\boldsymbol{L}})$. This, in turn, leads to the conclusion that

$$g^*(\mathscr{U},\gamma;\bar{\boldsymbol{L}})\leqslant g(\mathscr{U};\bar{\boldsymbol{L}}). \tag{3.8}$$

Further, the condition (3.7) when combined with (I, eqns (5.10) and (5.11)) implies that in the elastic range

$$\dot{\boldsymbol{G}}_{\mathrm{p}}=0\Leftrightarrow g(\mathscr{U};\bar{\boldsymbol{L}})\leqslant 0. \tag{3.9}$$

The proof (by contradiction) of the above result is as follows: Suppose that (*a*) $g(\mathscr{U};\bar{\boldsymbol{L}})<0$ implies $\dot{\boldsymbol{G}}_{\mathrm{p}}\neq 0$. Then, there must exist a value γ such that (*b*) $g^*(\mathscr{U},\gamma;\bar{\boldsymbol{L}})=0$. But, for the given values of $(\mathscr{U};\bar{\boldsymbol{L}})$, from (*a*) and (*b*) we have $g(\mathscr{U};\bar{\boldsymbol{L}})<g^*(\mathscr{U},\gamma;\bar{\boldsymbol{L}})$ which contradicts (3.8) and the proof is complete. The difference between (3.9) and the earlier result (I, eqns (5.10) and (5.11)) should be noted; the earlier result in Part I holds only in forward direction, while (3.9) is valid both ways. Once $(3.5)_2$ has been solved for γ, we may return to (3.2) and solve for $\boldsymbol{\rho}$ to obtain

$$\boldsymbol{\rho}=\hat{\boldsymbol{\rho}}(\mathscr{U};\bar{\boldsymbol{L}}). \tag{3.10}$$

Summarizing the above procedure for calculation of $\dot{\boldsymbol{G}}_{\mathrm{p}}$, we have

$$\left.\begin{aligned}(a)\quad & g(\mathscr{U};\bar{\boldsymbol{L}})\leqslant 0\Leftrightarrow \dot{\boldsymbol{G}}_{\mathrm{p}}=\boldsymbol{0},\\ (b)\quad & g(\mathscr{U};\bar{\boldsymbol{L}})>0\Leftrightarrow \dot{\boldsymbol{G}}_{\mathrm{p}}=\gamma\boldsymbol{\rho}(\mathscr{U};\bar{\boldsymbol{L}}),\end{aligned}\right\} \tag{3.11}$$

where γ is determined from $(3.5)_2$. The structure of the results (3.11) is similar to a constitutive equation for plastic strain rate in the usual formulation of plasticity or viscoplasticity (Casey & Naghdi 1984; Naghdi 1984*a*, *b*), although no direct contact can be made owing to the skew-symmetric part of $\dot{\boldsymbol{G}}_{\mathrm{p}}$ and related structure in the present theory.

As is customary in the standard developments of plasticity, we assume that the stress constitutive equation (I, eqn $(6.12)_1$) in the form $\boldsymbol{S}=\hat{\boldsymbol{S}}(\boldsymbol{E},\mathscr{W})$ for fixed values of the variables $\mathscr{W}$ is invertible so that we may write

$$\hat{\boldsymbol{E}}=\boldsymbol{E}(\boldsymbol{S},\mathscr{W}), \tag{3.12}$$

where $\mathscr{W}$ is defined by (I, eqn $(5.6)_2$). Then, any function of the variables $(\boldsymbol{E},\mathscr{W},\gamma)$ may be replaced by a different function of $(\boldsymbol{S},\mathscr{W},\gamma)$. In particular, the function g^* in (3.5) with the use of (3.12) can be expressed as

$$\begin{aligned}g^*(\boldsymbol{E},\mathscr{W},\gamma;\bar{\boldsymbol{L}})&=g^*(\hat{\boldsymbol{E}}(\boldsymbol{S},\mathscr{W}),\mathscr{W},\gamma;\bar{\boldsymbol{L}})\\ &=f^*(\boldsymbol{S},\mathscr{W},\gamma;\bar{\boldsymbol{L}}),\ \text{say}.\end{aligned} \tag{3.13}$$

Recalling $(2.21)_3$ and using the notation $f^*(\boldsymbol{S},\mathscr{W},0)=f(S,\mathscr{W})$, the equation

$$f(\boldsymbol{S},\mathscr{W};\bar{\boldsymbol{L}})=0 \tag{3.14}$$

for fixed values of $(\mathscr{W}, \bar{L})$ represents a smooth closed orientable hypersurface in the six-dimensional S-space and is called the yield surface in stress space. In view of (3.14), the results corresponding to (3.11) in $\boldsymbol{S}$-space are given by

$$\left.\begin{aligned} &(a) \quad f(\boldsymbol{S}, \mathscr{W}; \bar{\boldsymbol{L}}) \leqslant 0 \Rightarrow \dot{\boldsymbol{G}}_{\mathrm{p}} = 0, \\ &(b) \quad f(\boldsymbol{S}, \mathscr{W}; \bar{\boldsymbol{L}}) > 0 \Rightarrow \dot{\boldsymbol{G}}_{\mathrm{p}} = \hat{\boldsymbol{\rho}}(\mathscr{U}; \bar{\boldsymbol{L}}). \end{aligned}\right\} \tag{3.15}$$

In general, both (3.11) and (3.15) must be satisfied in the solution of a specific problem, but either one could be utilized at first depending on whether it is more convenient to introduce the yield function in strain or stress space. However, the basic loading criteria still remain in strain space as discussed in §6 of Part I.

In the remainder of this section, we illustrate the above procedure with reference to the special constitutive equations developed in §2. First, we note that the reduced balance law (3.1) in the absence of $\bar{\boldsymbol{L}}$ after also using $(2.16)_2$, (2.17) and $(2.18)_{1,2}$ becomes

$$_{\ell}\boldsymbol{C}\boldsymbol{G}_{\mathrm{p}}(-\boldsymbol{S}+\kappa\boldsymbol{\rho}+\boldsymbol{\beta}+\mu\gamma\boldsymbol{\rho}) = \boldsymbol{0}, \tag{3.16}$$

where the coefficients κ and μ were defined earlier in §2 (following (2.19)). The constitutive equation for the stress $\boldsymbol{S}$ can be displayed as

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{E}, \boldsymbol{G}_{\mathrm{p}}) = \bar{\boldsymbol{S}}(_{\ell}\boldsymbol{E}), \tag{3.17}$$

where $\boldsymbol{S}$ is now a different function from that in $(2.6)_2$. Then, remembering the procedure noted in the preceding paragraph (following (3.12)) for obtaining a yield function in stress space from a corresponding one in strain space, a von Mises type condition in the six-dimensional space of the strain after using (3.17) for the stress $\boldsymbol{S}$ can be written as a special case of (I, eqn (5.11)), namely

$$\|\boldsymbol{S}-\boldsymbol{\beta}\| = \kappa. \tag{3.18}$$

Utilizing the invertibility conditions for $_{\ell}\boldsymbol{C}$ and $\boldsymbol{G}_{\mathrm{p}}$, (3.16) can be rewritten in the form

$$(\kappa+\mu\gamma)\boldsymbol{\rho} = \boldsymbol{S}-\boldsymbol{\beta}. \tag{3.19}$$

Then, the inner product of (3.19) with itself after using also the fact that $\boldsymbol{\rho}$ is a unit tensor yields

$$\|\boldsymbol{S}-\boldsymbol{\beta}\|^2 = (\kappa+\mu\gamma)^2, \tag{3.20}$$

which is an equation for the determination of γ and corresponds to an equation of the form $f^*(\boldsymbol{S}, \kappa, \gamma) = 0$ in stress space. Solving (3.20) for γ we obtain

$$\kappa+\mu\gamma = \|\boldsymbol{S}-\boldsymbol{\beta}\| \quad \text{or} \quad \gamma = \mu^{-1}(\|\boldsymbol{S}-\boldsymbol{\beta}\|-\kappa). \tag{3.21}$$

Since $\gamma > 0$, it is easily seen that the solution (3.21) is valid as long as

$$f(\boldsymbol{S}, \kappa) = \|\boldsymbol{S}-\boldsymbol{\beta}\|-\kappa > 0. \tag{3.22}$$

Returning once more to (3.19), we may solve for $\boldsymbol{\rho}$ and obtain

$$\boldsymbol{\rho} = (\boldsymbol{S}-\boldsymbol{\beta})/\|\boldsymbol{S}-\boldsymbol{\beta}\|, \tag{3.23}$$

where in writing (3.23) we have also used (3.20) consistent with the fact that $\boldsymbol{\rho}$ is a unit tensor. Next, from a combination of (3.21)–(3.23) and (3.15) follow the criteria

$$\left.\begin{aligned} &\|\boldsymbol{S}-\boldsymbol{\beta}\|-\kappa \leqslant 0 \Rightarrow \dot{\boldsymbol{G}}_{\mathrm{p}} = 0, \\ &\|\boldsymbol{S}-\boldsymbol{\beta}\|-\kappa > 0 \Rightarrow \dot{\boldsymbol{G}}_{\mathrm{p}} = \gamma(\boldsymbol{S}-\boldsymbol{\beta})/\|\boldsymbol{S}-\boldsymbol{\beta}\| = \tilde{\psi} \text{ (say)}, \end{aligned}\right\} \tag{3.24}$$

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with the function $\tilde{\psi}$ given by

$$\tilde{\psi} = \gamma/\|\boldsymbol{S}-\boldsymbol{\beta}\| = \mu^{-1}(1-\kappa/\|\boldsymbol{S}-\boldsymbol{\beta}\|). \tag{3.25}$$

It can be easily observed from (3.25) that when $\|\boldsymbol{S}-\boldsymbol{\beta}\| \gg \kappa$, $\tilde{\psi} \to \mu^{-1}$ so that in this special case deformation is governed mainly by a purely visous response. In the context of metal plasticity in an isothermal environment, the condition $\|\boldsymbol{S}-\boldsymbol{\beta}\| \gg \kappa$ does not arise. (As is well known, a visous response beomes significant at high temperatures. Such conditions require considerations of thermomechanical effects which are outside the scope of the present paper.) Indeed for most metals at constant temperatures, we may neglect the viscous response entirely so that in the absence of viscous effect (3.19) reduces to

$$\kappa\boldsymbol{\rho} = \boldsymbol{S}-\boldsymbol{\beta}. \tag{3.26}$$

Solving (3.26) for the unit tensor $\boldsymbol{\rho}$, we again obtain the expression (3.23) provided that

$$\|\boldsymbol{S}-\boldsymbol{\beta}\| = \kappa. \tag{3.27}$$

Moreover, since (3.26) does not involve the magnitude γ of the rate of $\boldsymbol{G}_{\mathrm{p}}$, for the simple case of the yield function (3.20) subject to the restriction $\|\boldsymbol{S}-\boldsymbol{\beta}\| = \kappa$ one cannot obtain γ as a function of the basic kinematical variables.

The foregoing special example clearly demonstrates that the procedure for the determination of $\dot{\boldsymbol{G}}_{\mathrm{p}}$ must be modified when the rate-dependent response term $\hat{\boldsymbol{K}}_3$ (see I, eqn (6.18)) is suppressed. This issue will be further discussed in the next section.

4. A special rate-independent theory

We consider here a special case of the rate-dependent theory of §3, which in the spirit of classical developments on the subject may be identified as a rate-independent theory. First, however, we recall that in a standard development of the (rate-independent) theory of elastic-plastic materials, constitutive equations such as those for the rate of plastic strain and the rate of hardening are expressed as linear functions of the rate of strain with coefficient response functions in these equations, as well as the stress response function and the yield function, being independent of the rate of strain (or the rate of stress) and time derivatives of other independent variables of the theory (for further background, see Naghdi 1990). As will become evident presently, the special theory discussed here is somewhat analogous to the usual (rate-independent) theory of elastic-plastic materials. However, the *loading* conditions are different than in the usual classical formulations due to the presence here of the gradient of the plastic deformation (see §6*c* of Part I). Indeed, the special theory of this section becomes analogous to the usual plasticity when the dependence upon the gradient of the plastic deformation (which represents a measure of the dislocation density as described in §3*c* of Part I) is suppressed.

Proceeding with our main task, we first suppress the viscoplastic response from $\hat{\boldsymbol{K}}_3$ in $\hat{\boldsymbol{K}}$ given by (I, eqn (6.18)) and also neglect the effect of the contact stress ${}_{\mathrm{R}}\mathscr{M}$. Then, in the absence of body force $\boldsymbol{L}$, the reduced balance law (3.1) takes the simple form

$$\hat{\boldsymbol{K}}_2(\mathscr{U},\boldsymbol{\rho}) = -\hat{\boldsymbol{K}}_1(\mathscr{U}), \quad (\dot{\boldsymbol{G}}_{\mathrm{p}} \neq 0). \tag{4.1}$$

It follows from the above result that during plastic deformation, the elastic part $\hat{\boldsymbol{K}}_1$ of the response function $\hat{\boldsymbol{K}}$ is exactly balanced by the rate-type term $\hat{\boldsymbol{K}}_2$. Moreover, when the material response is purely elastic ($\dot{\boldsymbol{G}}_{\mathrm{p}} = \boldsymbol{0}$) and $\hat{\boldsymbol{K}} = \hat{\boldsymbol{K}}_1$, the function $\hat{\boldsymbol{K}}_1$ equals $-(\boldsymbol{K})_{\mathrm{ind}}$ as can be seen from (I, eqns (5.5) and $(6.7)_1$). Clearly then, the rate-type term $\hat{\boldsymbol{K}}_2$ during plastic deformation and the indeterminate $({}_{\mathrm{R}}\boldsymbol{K})_{\mathrm{ind}}$ during elastic deformation are *both* determined by the same elastic response function $\hat{\boldsymbol{K}}_1(\mathscr{U})$. Further, from the discussion of constraint in §5 of Part I and the result (3.9), it follows that as long as the condition

$$\Phi(-\hat{\boldsymbol{K}}_1(\mathscr{U}), \mathscr{U}) = g(\mathscr{U}) < 0 \tag{4.2}$$

is satisfied, the material response is elastic and $\dot{\boldsymbol{G}}_{\mathrm{p}} = \boldsymbol{0}$.

In the presence of plastic deformation, (4.1) provides nine scalar equations for eight components of the unit tensor $\boldsymbol{\rho}$. Clearly, the algebraic equations (4.1) for the determination of $\boldsymbol{\rho}$ have no solution for arbitrary values of the variables $\mathscr{U}$ defined by (I, eqn $(5.6)_1$). However, from the discussion in §6*b* of Part I pertaining to the geometrical interpretation of $\hat{\boldsymbol{K}}_2$, we recall that the values of $\hat{\boldsymbol{K}}_2$ for given values of $\mathscr{U}$ always lie on the surface Φ. This, in turn, implies that in the presence of plastic deformation we must have

$$\Phi(\hat{\boldsymbol{K}}_2(\mathscr{U}, \boldsymbol{\rho}), \mathscr{U}) = \Phi(-\hat{\boldsymbol{K}}_1(\mathscr{U}), \mathscr{U}) = g(\mathscr{U}) = 0. \tag{4.3}$$

We now proceed to show that the above equation, which is similar to (3.4) after suppressing the rate term in the function $\hat{\boldsymbol{K}}^*$ defined by (3.3), provides both the necessary and sufficient condition for the inversion of (4.1) to obtain $\boldsymbol{\rho}$. To see that (4.3) is necessary, we note that if (4.1) can be solved for $\boldsymbol{\rho}$, then by virtue of (4.1) and (I, eqns (6.23) and (6.32)), we must have

$$\Phi(\boldsymbol{K}_2, \mathscr{U}) = \Phi(\hat{\boldsymbol{K}}_2(\mathscr{U}, \boldsymbol{\rho}), \mathscr{U}) = \Phi(-\hat{\boldsymbol{K}}_1(\mathscr{U}), \mathscr{U}) = g(\mathscr{U}) = 0, \tag{4.4}$$

which proves the necessity. To prove sufficiency, we observe that if (4.3) is satisfied for given values of $\mathscr{U}$, then the value of $\hat{\boldsymbol{K}}_2$ calculated from (4.1) always lies on the surface $\partial\mathscr{K}_2$ in $\boldsymbol{K}_2$-space; and, in line with earlier remarks in the paragraph immediately following (I, eqn (6.22), $\boldsymbol{K}_2$ lies in the range of the function $\hat{\boldsymbol{K}}_2$ and hence there exists at least one value of $\boldsymbol{\rho}$ for which (4.1) is satisfied.

Thus, as long as (4.3) is satisfied, we may obtain a (possibly non-unique) value of $\boldsymbol{\rho}$ by inverting (4.1). It is clear that (4.1) cannot supply the magnitude γ of $\dot{\boldsymbol{G}}_{\mathrm{p}}$; however, as discussed below, γ can be determined from the condition (4.3).

(*a*) *An illustration of the procedure for the determination of the rate of plastic deformation*

We now illustrate the procedure discussed earlier in this section for the determination of $\boldsymbol{\rho}$ with the use of the special constitutive equations of §2. With the help of $(2.16)_2$, (2.17) and $(2.18)_{1,2}$, the reduced balance law (4.1) can be rewritten in the form

$${}_{\ell}\boldsymbol{C}\boldsymbol{G}_{\mathrm{p}}(\kappa\boldsymbol{\rho}+\boldsymbol{\beta}) = {}_{\ell}\boldsymbol{C}\boldsymbol{G}_{\mathrm{p}}\,\boldsymbol{S} \tag{4.5}$$

while the yield function (I, eqn (5.11)) will be taken to be the same as (3.18). Using the invertibility conditions of ${}_{\ell}\boldsymbol{C}$ and $\boldsymbol{G}_{\mathrm{p}}$, (4.5) can be solved for $\kappa\boldsymbol{\rho}$ to obtain

$$\kappa\boldsymbol{\rho} = \boldsymbol{S}-\boldsymbol{\beta}, \tag{4.6}$$

which is the special case of (3.19) after the neglect of the viscous term.

The expression (4.6) at once can be solved for $\boldsymbol{\rho}$ in the form

$$\boldsymbol{\rho} = (\boldsymbol{S}-\boldsymbol{\beta})/\|\boldsymbol{S}-\boldsymbol{\beta}\| = \kappa^{-1}(\boldsymbol{S}-\boldsymbol{\beta}), \tag{4.7}$$

provided that the stress $\boldsymbol{S}$ satisfies (3.18). Having obtained the result (4.7), the rate of plastic deformation can be displayed as

$$\dot{\boldsymbol{G}}_{\mathrm{p}} = \gamma \boldsymbol{\rho} = (\gamma/\kappa)(\boldsymbol{S}-\boldsymbol{\beta}) = \phi(\boldsymbol{S}-\boldsymbol{\beta}), \quad \text{say}, \tag{4.8}$$

where the scalar ϕ is as yet unknown. While equations (3.18) and (4.8) are formally analogous to the corresponding equations in the well-known Prandtl–Reuss equations (see §5B in Naghdi 1990), the procedure for the determination of the scalar function ϕ is complicated by the presence of the gradient of $\boldsymbol{G}_{\mathrm{p}}$ and will be discussed next.

(b) The loading criteria and the determination of γ in the rate-independent theory

In the example discussed in §3 in the context of the rate-dependent theory, the conditions for loading resulted in a simple criterion that had to be satisfied at every material point of the body. In the present discussion of the rate-independent theory the situation is quite different: This is because in the rate-dependent theory of §3 once the condition for initiation of plastic deformation was satisfied at a particular instant of time, further evolution of plastic deformation is chiefly governed by the general balance laws (I, eqn (4.12)). By contrast, in the special development of this section the reduced balance laws are degenerate; and this necessitates that the condition for plastic deformation, i.e. the loading criteria, must be checked at every instant of time. The latter statement needs further elaboration: It is evident from (I, eqn (6.2)) that the conditions

$$\left.\begin{aligned} &\text{(i)} \quad \gamma > 0 \Rightarrow \dot{\boldsymbol{G}}_{\mathrm{p}} \neq \mathbf{0}, \\ &\text{(ii)} \quad \gamma = 0 \Rightarrow \dot{\boldsymbol{G}}_{\mathrm{p}} = \mathbf{0} \quad \text{(or the elastic range).} \end{aligned}\right\} \tag{4.9}$$

Thus, if one is able to determine the magnitude γ of the rate of plastic deformation in terms of the purely kinematical variables of the theory, then the nature of the presence or absence of the rate of plastic deformation would be clear from (4.9). However, results of this kind are not easily realizable in the rate-independent theory and we need to outline a procedure for obtaining γ.

Consider now the state of deformable body $\mathscr{B}$ at some instant of time t. If at some material point which occupies the place $\boldsymbol{x}$ at time t the condition $g(\mathscr{U}) < 0$ is met, then $\boldsymbol{x}$ is in an elastic state and $\dot{\boldsymbol{G}}_{\mathrm{p}} = \mathbf{0}$. Thus, for the purpose of establishing the loading criteria in the presence of plastic deformation, it will suffice to focus attention only on an arbitrary material volume at time t which in the reference configuration occupies the region of space $\mathscr{P}_0^{\mathrm{p}}$ with boundary surface $\partial\mathscr{P}_0^{\mathrm{p}}$ where $g(\mathscr{U}) = 0$. Now, let us assume for the moment that each material point in $\mathscr{P}_0^{\mathrm{p}}$ is undergoing plastic deformation. Then, since (4.3) is satisfied, the material derivative of $g(\mathscr{U})$ is

$$\dot{g}(\mathscr{U}) = \frac{\partial g}{\partial \boldsymbol{E}} \cdot \dot{\boldsymbol{E}} + \frac{\partial g}{\partial \boldsymbol{G}_{\mathrm{p}}} \cdot \dot{\boldsymbol{G}}_{\mathrm{p}} + \frac{\partial g}{\partial(\operatorname{grad} \boldsymbol{G}_{\mathrm{p}})} \cdot \overline{\operatorname{Grad} \boldsymbol{G}_{\mathrm{p}}}^{\boldsymbol{\cdot}} = 0. \tag{4.10}$$

Substituting $\dot{\boldsymbol{G}}_{\mathrm{p}} = \gamma\boldsymbol{\rho}$, where the unit tensor $\boldsymbol{\rho}$ is determined from the inversion of (4.1), $(4.10)_2$ may be rewritten as

$$\hat{g} + \boldsymbol{a} \cdot \operatorname{grad} \gamma + A\gamma = 0, \tag{4.11}$$

where $\hat{g}$ is defined by

$$\hat{g} = \frac{\partial g}{\partial \boldsymbol{E}} \cdot \dot{\boldsymbol{E}}, \tag{4.12}$$

the vector-valued coefficient $\boldsymbol{a}$ and the scalar-valued A are given by

$$\boldsymbol{a} = \left[\frac{\partial g}{\partial G^{\mathrm{p}}_{AB,\,C}}\rho_{AB}\right]\boldsymbol{E}_C, \quad A = \left[\frac{\partial g}{\partial G^{\mathrm{p}}_{AB,\,C}}\rho_{AB,\,C} + \frac{\partial g}{\partial G^{\mathrm{p}}_{AB}}\rho_{AB}\right] \tag{4.13}$$

and where a comma following a subscript in (4.13) indicates partial differentiation.

The form (4.11) represents a first-order partial differential equation for γ and may be solved by using the method of characteristics (see Zauderer 1989, pp. 46–62). Before outlining the procedure for solving (4.11), however, we observe that for the special case in which $\boldsymbol{a} = \boldsymbol{0}$, (4.11) reduces to an algebraic equation for γ which may be solved to yield

$$\gamma = -\hat{g}/A. \tag{4.14}$$

Since γ (the magnitude of the tensor $\dot{\boldsymbol{G}}_{\mathrm{p}}$) must be positive and since we expect $\hat{g}$ to be positive, it follows that A must be negative in order that (4.11) be meaningful. Thus we assume $\hat{g} > 0$ during plastic deformation, while $\hat{g} \leqslant 0$ in the absence of plastic deformation. In summary, if $\boldsymbol{a} = \boldsymbol{0}$ at a material point, we have

$$\left.\begin{aligned} g<0 \quad \text{or} \quad g=0 \quad &\text{and} \quad \hat{g} \leqslant 0 \Rightarrow \gamma = 0, \\ g=0 \quad &\text{and} \quad \hat{g} > 0 \Rightarrow \gamma = -\hat{g}/A. \end{aligned}\right\} \tag{4.15}$$

Thus, at all points where the vector-valued coefficients $\boldsymbol{a}$ vanishes, the conditions (4.15) are formally identical to the loading criteria of the strain-space formulation of plasticity and the assumed constitutive expression for the rate of plastic strain has the same form as $\dot{\boldsymbol{G}}_{\mathrm{p}} = -(\hat{g}/A)\boldsymbol{\rho}$ (see §5 of Naghdi 1990).

Returning to the general case in which $\boldsymbol{a} \neq \boldsymbol{0}$, we consider the referential (or lagrangian) form of the characteristic base curves $\boldsymbol{X}(\lambda)$ parametrized by λ, which are solutions to the ordinary differential equation (Zauderer 1989, p. 49):

$$\mathrm{d}\boldsymbol{X}/\mathrm{d}\lambda = -\boldsymbol{a}(\boldsymbol{X}). \tag{4.16}$$

The existence and uniqueness theorems of ordinary differential equations, stipulating certain smoothness assumptions for $\boldsymbol{a}$ (see Cartan 1983, pp. 110–112), guarantees that exactly one solution curve passes through a given material point in $\mathscr{P}^{\mathrm{p}}_0$. With the use of the chain rule, it is seen that along such a curve

$$\boldsymbol{a}\cdot\operatorname{grad}\gamma = -\mathrm{d}\gamma/\mathrm{d}\lambda, \tag{4.17}$$

so that (4.11) may be recast in the form

$$\mathrm{d}\gamma/\mathrm{d}\lambda - A(\lambda)\,\gamma = \hat{g}(\lambda), \tag{4.18}$$

where the dependence of A and $\hat{g}$ on λ is explicitly displayed.

The ordinary differential equation (4.18) admits an integrating factor of the form $\exp(-\int A(u)\,\mathrm{d}u)$. Using this integrating factor, (4.18) may be solved along any given characteristic curve to yield

$$\gamma(\lambda) = \left[\exp\int_{\lambda_0}^{\lambda} A(u)\,\mathrm{d}u\right]\int_{\lambda_0}^{\lambda}\left\{\hat{g}(u)\left[\exp\int_{\lambda_0}^{u} -A(\bar{u})\,\mathrm{d}\bar{u}\right]\mathrm{d}u\right\} + \gamma(\lambda_0), \tag{4.19}$$

where λ_0 is some value of the parameter λ along the curve and $\gamma(\lambda_0)$ is the initial value of γ which has to be specified. With reference to (4.19), it should be noted that since $\boldsymbol{a}$ depends on $\operatorname{grad}\boldsymbol{G}_{\mathrm{p}}$ by virtue of (4.13), the solution for γ given by (4.19) necessarily involves integral of $\hat{g}$ (through (4.11)).

The solution (4.19) is applicable only to the segment of the curve which lies within the part $\mathscr{P}_0^{\mathrm{p}}$; and is, of course, meaningful only on those points where $\gamma(\lambda) > 0$. All other material points for which the right-hand side of (4.19) becomes zero or negative correspond, respectively, to *neutral loading* and *unloading*. The points on $\boldsymbol{x}(\lambda)$ which correspond to neutral loading and unloading depend on the value assigned to $\gamma(\lambda_0)$ and require further elaboration. To this end, consider a particular characteristic curve whose orientation is in the direction of $-\boldsymbol{a}$ (on the right-hand side of (4.16)) and is parametrized by λ such that $\lambda = 0$ corresponds to $\boldsymbol{X}(0) \in \partial\mathscr{P}_0^{\mathrm{p}}$ where the curve first enters $\mathscr{P}_0^{\mathrm{p}}$. Now, let the curve be traversed in the direction $-\boldsymbol{a}$ and specify the value

$$\gamma(\lambda_0) = \hat{g}(\lambda_0)/-A(\lambda_0) \tag{4.20}$$

at the first point on the curve where $\hat{g} > 0$. In conformity with (4.20), we also stipulate that $\gamma(\lambda) = 0$ for all $\lambda \in [0, \lambda_0)$ since $\hat{g}(\lambda) \leqslant 0$ for $\lambda \in [0, \lambda_0)$, so that points on the curve $\boldsymbol{X}(\lambda)$, $\lambda \in [0, \lambda_0)$ are either in the state of neutral loading (corresponding to $\gamma = 0$, $\hat{g} = 0$) or in the state of unloading (corresponding to $\gamma = 0$, $\hat{g} < 0$).

Since $\gamma(\lambda_0) > 0$ in the presence of plastic deformation, we may use (4.19) for $\lambda > \lambda_0$ to determine γ as long as the right-hand side of (4.19) is positive. Suppose now that the right-hand side of (4.19) becomes equal to zero at some point $\lambda_1 > \lambda_0$. Then, at this point $\gamma(\lambda_1) = 0$ and (since $\gamma(\lambda < \lambda_1) > 0$) by (4.18) we must have $\hat{g}(\lambda_1) < 0$. Next, suppose that for points $\boldsymbol{X}(\lambda)$ in the range $\lambda_1 \leqslant \lambda < \lambda_2$, $\hat{g}(\lambda) \leqslant 0$, then $\gamma(\lambda) = 0$ and $\hat{g}(\lambda_2) > 0$ and at $\lambda = \lambda_2$ specify the value

$$\gamma(\lambda_2) = \hat{g}(\lambda_2)/-A(\lambda_2). \tag{4.21}$$

Then, again (4.19) is valid for $\lambda > \lambda_2$ and the entire procedure may be repeated until a point on the boundary $\partial\mathscr{P}_0^{\mathrm{p}}$ is reached. This procedure can be repeated for every characteristic curve in $\mathscr{P}_0^{\mathrm{p}}$ and the value of γ can be determined for every point in $\mathscr{P}_0^{\mathrm{p}}$.

To summarize the above procedure for the determination of γ, let $\boldsymbol{X}(\lambda)$ be any characteristic base curve of (4.11) which enters the region $\mathscr{P}_0^{\mathrm{p}}$ at $\lambda = 0$ and leaves it at $\lambda = \bar{\lambda}$. Further, let the curve $\boldsymbol{X}(\lambda)$ be divided into the following segments:

(1) absence of plastic deformation with $\lambda \in [0, \lambda_0)$ so that

$$g(\lambda) = 0, \quad \hat{g}(\lambda) < 0 \Rightarrow \gamma = 0 \quad \text{(unloading)},$$
$$g(\lambda) = 0, \quad \hat{g}(\lambda) = 0 \Rightarrow \gamma = 0 \quad \text{(neutral loading)};$$

(2) the first occurrence of plastic deformation (initiation of yield) at $\lambda = \lambda_0$ with $\hat{g}(\lambda_0) > 0$ and with $\gamma(\lambda_0) > 0$ given by (4.20);

(3) presence of plastic deformation with $\lambda \in [\lambda_0, \lambda_1)$ representing a segment of the curve where (4.19) which involves integral of $\hat{g}$ is valid and

$$\gamma(\lambda) = \text{RHS of (4.19)} > 0 \quad \text{(loading)};$$

(4) absence of plastic deformation with $\lambda \in [\lambda_1, \lambda_2)$ so that

$$g(\lambda) \leqslant 0 \Rightarrow \gamma = 0 \quad \text{(same as step (1))};$$

(5) another occurrence of plastic deformation at $\lambda = \lambda_2$ with $\hat{g}(\lambda_2) > 0$ and with $\gamma(\lambda_2) > 0$ given by (4.21).

The steps (3) to (5) can be repeated for $\lambda_2 < \lambda \leqslant \bar{\lambda}$, and so on. This completes the discussion of the loading criteria and the determination of γ in the rate-independent theory.

Before closing this section, it should be emphasized that after suppressing the dependence of the yield function on grad $\boldsymbol{G}_{\mathrm{p}}$, the equation for the determination of the rate of plastic deformation $\dot{\boldsymbol{G}}_{\mathrm{p}}$ (apart from the presence of the skew-symmetric part of $\boldsymbol{G}_{\mathrm{p}}$) is formally analogous to that of the usual (rate-independent) theory of plasticity.

5. Infinitesimal theory

It is of interest to include here a brief account of the infinitesimal theory which results from linearization of the basic kinematics of Part I, as well as the balance laws and the constitutive equations of §2. We begin by first introducing the second-order tensors representing the relative deformation gradient $\boldsymbol{H}$, the relative lattice deformation tensor ${}_{\ell}\boldsymbol{H}$ and the relative plastic deformation tensor $\boldsymbol{H}_{\mathrm{p}}$ by

$$\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{I}, \quad {}_{\ell}\boldsymbol{H} = {}_{\ell}\boldsymbol{F} - \boldsymbol{I}, \quad \boldsymbol{H}_{\mathrm{p}} = \boldsymbol{G}_{\mathrm{p}} - \boldsymbol{I}, \tag{5.1}$$

where the tensors $\boldsymbol{F}$, ${}_{\ell}\boldsymbol{F}$ and $\boldsymbol{G}_{\mathrm{p}}$ are defined by (I, eqns $(3.2)_1$, $(3.6)_1$, (3.8)) and $\boldsymbol{I}$ is the identity tensor. In view of (I, eqn (3.8)), the truth of the following relationship between $\boldsymbol{H}$, ${}_{\ell}\boldsymbol{H}$ and $\boldsymbol{H}_{\mathrm{p}}$ can be easily verified:

$$\boldsymbol{H} = {}_{\ell}\boldsymbol{H} + \boldsymbol{H}_{\mathrm{p}} + {}_{\ell}\boldsymbol{H}\,\boldsymbol{H}_{\mathrm{p}}, \tag{5.2}$$

while the relative lagrangian strain $\boldsymbol{E}$ defined previously by $(3.26)_1$ can also be recorded in the form

$$\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^{\mathrm{T}} + \boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}). \tag{5.3}$$

Further, we introduce for later convenience the following definitions:

$$\boldsymbol{e} = \tfrac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^{\mathrm{T}}), \quad {}_{\ell}\boldsymbol{e} = \tfrac{1}{2}({}_{\ell}\boldsymbol{H} + {}_{\ell}\boldsymbol{H}^{\mathrm{T}}), \quad \boldsymbol{e}_{\mathrm{p}} = \tfrac{1}{2}(\boldsymbol{H}_{\mathrm{p}} + \boldsymbol{H}_{\mathrm{p}}^{\mathrm{T}}), \tag{5.4}$$

and

$$\boldsymbol{\omega} = \tfrac{1}{2}(\boldsymbol{H} - \boldsymbol{H}^{\mathrm{T}}), \quad {}_{\ell}\boldsymbol{\omega} = \tfrac{1}{2}({}_{\ell}\boldsymbol{H} - {}_{\ell}\boldsymbol{H}^{\mathrm{T}}), \quad \boldsymbol{\omega}_{\mathrm{p}} = \tfrac{1}{2}(\boldsymbol{H}_{\mathrm{p}} - \boldsymbol{H}_{\mathrm{p}}^{\mathrm{T}}). \tag{5.5}$$

(*a*) *Linearized version of the basic equations*

We now proceed to summarize an invariant form of the equations of the infinitesimal theory in the manner discussed in a different context by Casey & Naghdi (1985). Thus, let a measure of smallness ϵ be defined by

$$\epsilon = \max\,[\sup \|{}_{\ell}\boldsymbol{H}\|, \sup \|\boldsymbol{H}_{\mathrm{p}}\|], \tag{5.6}$$

where the norm of any second-order tensor $\boldsymbol{A}$ is defined by $\|\boldsymbol{A}\| = (\boldsymbol{A}\cdot\boldsymbol{A})^{\frac{1}{2}}$, and the supremum (sup) is taken over the region occupied by the body in its reference configuration. If $\boldsymbol{Z}$ is any tensor-valued function of $({}_{\ell}\boldsymbol{H}, {}_{\mathrm{p}}\boldsymbol{H})$ defined in a neighbourhood of $({}_{\ell}\boldsymbol{H}, \boldsymbol{H}_{\mathrm{p}}) = (\boldsymbol{0}, \boldsymbol{0})$ and satisfying the condition that there exists a non-negative real constant K such that

$$\|\boldsymbol{Z}\| = K\epsilon^{n} \quad \text{as} \quad \epsilon \to 0, \tag{5.7}$$

then we write

$$\boldsymbol{Z} = \boldsymbol{O}(\epsilon^{n}) \quad \text{as} \quad \epsilon \to 0, \tag{5.8}$$

n being a non-negative real number. We further assume that all referential gradients of $({}_{\ell}\boldsymbol{H}, \boldsymbol{H}_{\mathrm{p}})$ are of the same order as $({}_{\ell}\boldsymbol{H}, \boldsymbol{H}_{\mathrm{p}})$.

Now, in light of (5.6), from (5.2) we may deduce that

$$\boldsymbol{H} = {}_{\ell}\boldsymbol{H} + \boldsymbol{H}_{\mathrm{p}} + \boldsymbol{O}(\epsilon^{2}) \quad \text{as} \quad \epsilon \to 0. \tag{5.9}$$

In the same manner as that which led to (5.9), we may establish other kinematical

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results, but henceforth for ease of writing we do not display the statement 'as $\epsilon \to 0$'. Thus from (5.3) and (5.4) we obtain

$$\boldsymbol{E} = \boldsymbol{e} + \boldsymbol{O}(\epsilon^2) = {}_{\ell}\boldsymbol{e} + \boldsymbol{e}_{\mathrm{p}} + \boldsymbol{O}(\epsilon^2). \tag{5.10}$$

Similarly, we have

$$\left.\begin{aligned} J &= \det \boldsymbol{F} = 1 + e_{kk} + O(\epsilon^2), \\ {}_{\ell}J &= \det {}_{\ell}\boldsymbol{F} = 1 + e^{\ell}_{kk} + O(\epsilon^2), \\ J_{\mathrm{p}} &= \det(\boldsymbol{G}_{\mathrm{p}}) = 1 + e^{\mathrm{p}}_{kk} + O(\epsilon^2), \end{aligned}\right\} \tag{5.11}$$

and

$$\left.\begin{aligned} \boldsymbol{R} &= \boldsymbol{F}\boldsymbol{U}^{-1} = \boldsymbol{I} + \boldsymbol{\omega} = \boldsymbol{I} + {}_{\ell}\boldsymbol{\omega} + \boldsymbol{\omega}_{\mathrm{p}}, \\ \boldsymbol{U}_{\mathrm{p}} &= (\boldsymbol{G}_{\mathrm{p}}^{\mathrm{T}}\boldsymbol{G}_{\mathrm{p}})^{\frac{1}{2}} = \boldsymbol{I} + \boldsymbol{e}_{\mathrm{p}} + O(\epsilon^2), \\ \boldsymbol{R}_{\mathrm{p}} &= \boldsymbol{G}_{\mathrm{p}}\boldsymbol{U}_{\mathrm{p}}^{-1} = \boldsymbol{I} + \boldsymbol{\omega}_{\mathrm{p}}, \end{aligned}\right\} \tag{5.12}$$

where in (5.11) to the order of approximation $e_{kk} = \operatorname{tr} \boldsymbol{e}$ with similar results for e^{ℓ}_{kk} and e^{p}_{kk}. Also, the expression (I, eqn (3.49)) for the dislocation density when linearized can be shown to yield

$$\boldsymbol{\alpha} = -(\operatorname{curl} \boldsymbol{G}_{\mathrm{p}})^{\mathrm{T}} = -(\operatorname{curl} \boldsymbol{H}_{\mathrm{p}})^{\mathrm{T}} = (\operatorname{curl} {}_{\ell}\boldsymbol{H})^{\mathrm{T}} + O(\epsilon^2), \tag{5.13}$$

where in obtaining (5.13) we have made use of (5.4), as well as the fact that $\operatorname{curl} \boldsymbol{H} = \boldsymbol{0}$. The truth of the result (5.13) can be easily verified by using the component form of the tensor $\boldsymbol{\alpha}$ as stated in (I, following eqn (3.50)) and after using the component versions of $(5.1)_3$ and (5.9), i.e. $G^{\mathrm{p}}_{AK} = H^{\mathrm{p}}_{AK} + \delta_{AK}$ and $H^{\mathrm{p}}_{AK} = H_{iK}\,\delta_{iA} - H^{\ell}_{AK}$. Thus

$$\alpha_{AB} = \epsilon_{KBM}\, G^{\mathrm{p}}_{AK,M} = \epsilon_{KBM}\, H^{\mathrm{p}}_{AK} = \epsilon_{KBM}(H_{iK,M}\,\delta_{iA} - H^{\ell}_{AK,M}). \tag{5.14}$$

But the first term in $(5.14)_3$ with the help of $(5.1)_1$ yields

$$\delta_{iA}\,\epsilon_{KBM}\, H_{iK,M} = \delta_{iA}\,\epsilon_{KBM}\, x_{i,KM} = 0,$$

so that $(5.14)_3$ reduces to $\alpha_{AB} = -\epsilon_{KBM}\, H^{\ell}_{AK,M}$ as the component version of $(5.13)_3$.

The linearized versions of the balance laws can be obtained in the usual manner from (I, eqns $(4.9)_{1,2}$ and (4.12)). The linearized versions of (I, eqns $(4.9)_{1,2}$) for mass conservation and ordinary linear momentum are well known and need not be recorded here; however, we note that within the scope of the linearized theory the distinction between all three stress tensors $(\boldsymbol{P}, \boldsymbol{S}, \boldsymbol{T})$ disappears (correct to $\boldsymbol{O}(\epsilon^2)$ as $\epsilon \to 0$). The linearization of the balance of the director momenta (I, eqn (4.10)) is also straightforward but for later reference we comment only on the linearization of the inertia term on the left-hand side of (I, eqn (4.12)): In view of the fact that

$$\boldsymbol{G}_{\mathrm{p}} = \boldsymbol{H}_{\mathrm{p}} + \boldsymbol{I} = (\boldsymbol{e}_{\mathrm{p}} + \boldsymbol{\omega}_{\mathrm{p}}) + \boldsymbol{I} + \boldsymbol{O}(\epsilon^2)$$

by $(5.1)_3$, $(5.4)_3$ and $(5.5)_3$, it follows that to $\boldsymbol{O}(\epsilon^2)$ the LHS of (I, eqn (4.10) $=$ $\rho_0\,\partial/\partial t\,[\mathcal{Y}(\dot{\boldsymbol{e}}_{\mathrm{p}} + \dot{\boldsymbol{\omega}}_{\mathrm{p}})]$.

Having dealt with the kinematical quantities and the balance laws, we now turn to the rate-independent terms in the response functions; and, in particular, assume that the strain energy function $\bar{\psi}$ in (2.3) can be expanded in series form about $({}_{\ell}\boldsymbol{H}, \boldsymbol{H}_{\mathrm{p}}) = (\boldsymbol{0}, \boldsymbol{0})$ and write

$$\rho_0 \psi = (\mathscr{C}[{}_{\ell}\boldsymbol{e}]) \cdot {}_{\ell}\boldsymbol{e} + O(\epsilon^3), \tag{5.15}$$

where the infinitesimal measure of strain ${}_{\ell}\boldsymbol{e}$ is defined by $(5.4)_2$ and

$$\mathscr{C} = \mathscr{C}_{IJKL}\, \boldsymbol{E}_I \otimes \boldsymbol{E}_J \otimes \boldsymbol{E}_K \otimes \boldsymbol{E}_L \tag{5.16}$$

is a fourth-order positive definite symmetric tensor. Keeping in mind the results (5.10)–(5.12) with ψ in the form (5.15) and remembering that the distinction between the stresses $\boldsymbol{S}$, $\boldsymbol{P}$, $\boldsymbol{T}$ disappears in the infinitesimal theory, the constitutive equations (2.6)–(2.8) reduce to

$$\boldsymbol{T} = \boldsymbol{S} + O(\epsilon^2) = \boldsymbol{P} + O(\epsilon^2) = \mathscr{C}[{}_\ell\boldsymbol{e}] + O(\epsilon^2) \tag{5.17}$$

and by (2.17) the response $\hat{\boldsymbol{K}}_1$ becomes

$$\hat{\boldsymbol{K}}_1 = -\boldsymbol{T} + O(\epsilon^2). \tag{5.18}$$

Preliminary to the linearization of the rate-dependent terms $\hat{\boldsymbol{K}}_2$ and $\hat{\boldsymbol{K}}_3$ in the response function $\hat{\boldsymbol{K}}$ in (I, eqn (6.18)), we observe that to the order of approximation under discussion, the expressions for ${}_\ell\boldsymbol{C}$, $\boldsymbol{G}_\mathrm{p}$ defined by (I, eqns (3.25) and (3.8)) and for κ, $\boldsymbol{\beta}$ and ψ defined by $(2.19)_{1,2}$ and (5.15) are:

$$\begin{gathered} {}_\ell\boldsymbol{C} = \boldsymbol{I} + \boldsymbol{O}(\epsilon), \quad \boldsymbol{G}_\mathrm{p} = \boldsymbol{G}_\mathrm{p}^{-\mathrm{T}} = \boldsymbol{I} + \boldsymbol{O}(\epsilon), \\ \kappa = O(\epsilon), \quad \boldsymbol{b} = \boldsymbol{O}(\epsilon), \quad \psi = O(\epsilon^2). \end{gathered} \tag{5.19}$$

Then, since the unit tensor $\boldsymbol{\rho} = \boldsymbol{O}(1)$, it readily follows from $(2.18)_1$ that

$$\hat{\boldsymbol{K}}_2 = \kappa\boldsymbol{\rho} + \boldsymbol{\beta} + \boldsymbol{O}(\epsilon^2). \tag{5.20}$$

Similarly, since $\dot{\boldsymbol{G}}_\mathrm{p} = \boldsymbol{O}(\epsilon)$ by $(5.19)_2$, from $(2.18)_2$ we have

$$\hat{\boldsymbol{K}}_3 = \mu\dot{\boldsymbol{G}}_\mathrm{p} + \boldsymbol{O}(\epsilon^2). \tag{5.21}$$

It should be noted here that despite the approximate nature of κ and $\boldsymbol{\beta}$ in (5.19), the yield function in stress space retains its form (3.18) but with $\boldsymbol{S}$ replaced by $\boldsymbol{T}$.

(*b*) *Reduction to Prandtl–Reuss type equations*

It is instructive to illustrate here the nature of the simplifications that result as a consequence of linearization, especially since such linearized theories frequently serve a useful purpose and have had a long history in the development of plasticity. We carry out our objective here with a discussion of constitutive equations for small deformations of elastic-plastic material within the scope of the rate-independent theory of §4.

Thus, in view of (5.17), the expression for the unit tensor $\boldsymbol{\rho} = O(1)$ of the rate of plastic deformation $\dot{\boldsymbol{G}}_\mathrm{p}$ is

$$\boldsymbol{\rho} = \kappa^{-1}(\boldsymbol{T} - \boldsymbol{\beta}). \tag{5.22}$$

Using the definitions $(5.1)_3$ for $\boldsymbol{H}_\mathrm{p}$, as well as $(5.4)_3$ and $(5.5)_3$ for $\boldsymbol{e}_\mathrm{p}$ and $\boldsymbol{\omega}_\mathrm{p}$ and remembering that the tensor $\boldsymbol{T}$ is symmetric, the linearized version of (4.8) gives

$$\dot{\boldsymbol{e}}_\mathrm{p} = \phi(\boldsymbol{T} - \boldsymbol{\beta}_{(\mathrm{sym})}), \quad \dot{\boldsymbol{\omega}}_\mathrm{p} = -\phi\boldsymbol{\beta}_{(\mathrm{skew})}, \quad \phi = \gamma/\kappa. \tag{5.23}$$

In (5.23), the notations $\boldsymbol{\beta}_{(\mathrm{sym})}$ and $\boldsymbol{\beta}_{(\mathrm{skew})}$ stand for the symmetric and skew-symmetric parts of $\boldsymbol{\beta}$ and the coefficient ϕ given by $(5.23)_3$ is the same as that in (4.8). Since $\boldsymbol{\beta}$ and κ in (5.23) depend (in addition to $\boldsymbol{G}_\mathrm{p}$) also on grad $\boldsymbol{G}_\mathrm{p}$ (see $(2.19)_{1,2}$ and (I, eqn $(5.6)_2$)), they necessarily represent the manifestation of *microstructural effects* in the linearized theory arising from the lattice deformations and dislocation densities.

To provide some contact with known results in classical plasticity, we consider now a special case of (5.23) and the yield function by setting

$$\boldsymbol{\beta} = \boldsymbol{0}, \quad \kappa = \text{const.}, \tag{5.24}$$

and thus suppressing all microstructural effects. In this case, the linearized form of plastic deformation $\dot{\boldsymbol{G}}_{\mathrm{p}}$ coincides with the symmetric tensor $\dot{\boldsymbol{e}}_{\mathrm{p}}$, $\dot{\boldsymbol{\omega}}_{\mathrm{p}}$ vanishes and the linearized form of the yield function (2.22) reduces to

$$f = \boldsymbol{T} \cdot \boldsymbol{T} = \kappa^2 = \text{const.} \tag{5.25}$$

Further, with reference to the yield surface in strain space, it is easily seen that upon substitution for $\boldsymbol{T}$ from $(5.17)_3$ into (5.25) we have

$$g(\mathscr{U}) = \mathscr{C}[\boldsymbol{e}-\boldsymbol{e}_{\mathrm{p}}] \cdot \mathscr{C}[\boldsymbol{e}-\boldsymbol{e}_{\mathrm{p}}] - \kappa^2 = 0, \tag{5.26}$$

where now the argument for g refers to the linearized forms of the set of variables defined by (I, eqn $(5.6)_1$]. Since g in (5.26) is independent of the variables $\mathscr{W}$ defined by (I, eqn $(5.6)_2$), it can be readily demonstrated that the magnitude γ of $\dot{\boldsymbol{e}}_{\mathrm{p}}$ is now given by

$$\gamma = \kappa\hat{g}/2(\mathscr{C}[\boldsymbol{T}] \cdot \boldsymbol{T}), \tag{5.27}$$

where now $\hat{g}$ is

$$\hat{g} = \mathscr{C}[\boldsymbol{T}] \cdot \dot{\boldsymbol{e}}. \tag{5.28}$$

The system of equations consisting of (5.23) with $\boldsymbol{\beta} = \boldsymbol{0}$, $(5.17)_3$ and (5.25)–(5.28) has been derived under the conditions $(5.24)_{1,2}$ which represent the neglect of microstructural effects. The well-known Prandtl–Reuss equations follow from these by imposing the condition of plastic incompressibility and after specializing the fourth-order tensor coefficient $\mathscr{C}$ in $(5.17)_1$ and the results in (5.26)–(5.28) to that appropriate for a stress response which is isotropic in its reference configuration.

The results reported here were obtained in the course of research supported by the Solid Mechanics Program of the U.S. Office of Naval Research under contract N00014-90-J-1959, R & T 4324-436 with the University of California at Berkeley.

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Received 17 January 1992; accepted 29 March 1993